Appendix:

Proof of Proposition 1:
Bidder *i*'s (linear) expected utility from biding b_i is:

$$EU = p(b_i)(v_i - \hat{b}) + (1 - p(b_i))\frac{S-Q}{N-Q}v_i \qquad \text{(A1)}$$

where $p(\cdot)$ represents the probability that she wins the auction with $p'(b_i) \geq 0$ and $\hat{b}$ represents the expected price conditional on b_i being one of the winning bids. Rewrite expression (A1) as follows:

$$EU = p(b_i)\left[\left(1 - \frac{S-Q}{N-Q}\right)v_i - \hat{b}\right] + \frac{S-Q}{N-Q}v_i . \qquad \text{(A2)}$$

Note that the second term of expression (A2) is independent of b_i. Therefore choosing b_i to maximize (A2) is identical to choosing b_i to maximize the following expression:

$$EU* = p(b_i)[v_i * - \hat{b}] \qquad \text{(A3)}$$

where $v_i* = \left(1 - \frac{S-Q}{N-Q}\right)v_i$. Expression (A3) is of the same form of the problem faced by a bidder in a second-price auction; with probability $p(b_i)$ bidder *i* wins a unit of the good, worth v_i*, and pays a price equal to $\hat{b}$. In this more familiar setting, the Nash equilibrium bidding strategy has each bidder bidding her value. Here, the result follows:

$$b_{UH,N}(v_i) = v_i* = \left(1 - \frac{S-Q}{N-Q}\right)v_i . \qquad \text{(A4)}$$

Proof of Proposition 3:
Assume that bidder *i* believes all other bidders are using the increasing bid function $b(v_l)$ for $l \neq i$ where the other subscripts have been suppressed. Choosing b_i to maximize $F(\pi(b_i))U(v_i - b_i) + [1 - F(\pi(b_i))]\left\{\frac{S-Q}{N-Q}U(v_i) + \left[1 - \frac{S-Q}{N-Q}\right]U(0)\right\}$ yields the following first order condition:

$$f(\pi(b_i))\pi'(b_i)U(v_i - b_i) - U'(v_i - b_i)F(\pi(b_i)) - f(\pi(b_i))\pi'(b_i)[pU(v_i) + (1-p)U(0)] = 0 \qquad \text{(A5)}$$

where $p \equiv \frac{S-Q}{N-Q}$.

Substituting $\frac{1}{b'(v_i)}$ for $\pi'(b_i)$ and v_i for $\pi(b_i)$, the expression reduces to

$$\frac{f(v_i)}{b'(v_i)}U(v_i - b_i) - U'(v_i - b_i)F(v_i) - \frac{f(v_i)}{b'(v_i)}[pU(v_i) + (1-p)U(0)] = 0 . \qquad \text{(A6)}$$

Rearranging expression (A6) yields:

$$b'(v_i) = \frac{f(v_i)}{F(v_i)}\frac{U(v_i - b_i) - [pU(v_i) + (1-p)U(0)]}{U'(v_i - b_i)} \qquad \text{(A7)}$$

Note that expressions (A5), (A6), and (A7) characterize the first order condition for the equilibrium bid function under the discriminative hybrid regardless of the bidder's risk preferences. Now, assuming risk neutrality and rearranging terms, we have

$$f(v_i)v_i(1-p) = f(v_i)b_{DH,N}(v_i) + b'_{DH,N}(v_i)F(v_i) \qquad \text{(A8)}$$

because $U(0)=0$. Rewrite (A8) as

$$\frac{d}{dv_i}\left[F(v_i)b_{DH,N}(v_i)\right] = f(v_i)v_i(1-p). \qquad \text{(A9)}$$

Because $b_{DH,N}(0)=0$, the solution to the above is given by

$$b_{DH,N}(v_i) = (1-p)\frac{1}{F(v_i)}\int_0^{v_i} xf(x)dx. \qquad \text{(A10)}$$

In order to confirm that the solution to (A10) is the Nash equilibrium we must confirm that the bid function is indeed increasing in *v*. Solving expression (A8) for $b'_{DH,N}(v_i)$ yields:

$$b'_{DH,N}(v_i) = \frac{f(v_i)}{F(v_i)}\left[(1-p)v_i - b_{DH,N}(v_i)\right] \qquad \text{(A11)}$$

which is positive provided $(1-p)v_i > b_{DH,N}(v_i) = (1-p)\frac{1}{F(v_i)}\int_0^{v_i} xf(x)dx$.

A comparison of the equilibrium bid functions for the discriminative hybrid and the discriminative auction when the lottery is absent (Harris and Raviv's [1981] equation (11)) confirms this inequality. Following Harris and Raviv [1981], the equilibrium bid function (assuming risk neutrality) for bidder *i* facing a discriminative auction, denoted $b_{D,N}(v_i)$, is equal to

$$b_{D,N}(v_i) = \frac{1}{F(v_i)}\int_0^{v_i} xf(x)dx. \qquad \text{(A12)}$$

Substituting $b_{D,N}(v_i)$ into the expression for $b_{DH,N}(v_i)$ yields

$$b_{DH,N}(v_i) = (1-p)b_{D,N}(v_i) \qquad \text{(A13)}$$

which suggests that the presence of the lottery in the hybrid mechanism causes risk neutral bidders to shade their bids by the probability of losing the lottery, $(1-p) \equiv \left(1 - \frac{S-Q}{N-Q}\right)$. Harris and Raviv [1981] show that $b_{D,N}(v_i) < v_i$. Combining the results, we have the following inequality:

$$b_{DH,N}(v_i) = \left(1 - \frac{S-Q}{N-Q}\right)b_{D,N}(v_i) < \left(1 - \frac{S-Q}{N-Q}\right)v_i < v_i. \qquad \text{(A14)}$$

Proof of Proposition 5:
The first part of the proof proceeds by showing $b'_{DH,A}(\hat{v}) > b'_{DH,N}(\hat{v})$ where

$$\hat{v} = \min\left\{v > 0 \middle| b_{DH,A}(v) = b_{DH,N}(v) \geq 0\right\}.$$

For the remainder of the proof, all subscripts are suppressed. By strict concavity of *U*,

$U(v) < U(v-b) + bU'(v-b)$ which implies $\frac{U(v)-U(v-b)}{U'(v-b)} < b$ because $U'(v) > 0$.

Multiplying the inequality by *-1* and adding and subtracting *v* from the right hand side yields:

$$\frac{U(v-b)-U(v)}{U'(v-b)} > -b + v - v$$

Rearranging, we have:

$$\left[\frac{U(v-b)}{U'(v-b)} - (v-b)\right] - \left[\frac{U(v)}{U'(v-b)} - v\right] > 0 \qquad \text{(A15)}$$

By concavity of *U* and $U(0) = 0$, we know that $\left[\frac{U(v-b)}{U'(v-b)} - (v-b)\right] > 0$. When *b = 0*,

$\left[\frac{U(v)}{U'(v-b)} - v\right] > 0$ by strict concavity of *U*. This term approaches zero as *b* increases, and is

equal to zero at some $b*$. For $b > b*$, $\left[\frac{U(v)}{U'(v-b)} - v\right] < 0$.

Suppose $\hat{v} > 0$, where $\hat{v}$ is defined above, is such that $0 < b_{DH,A}(\hat{v}) = b_{DH,N}(\hat{v}) = b < b^*$, then

$\left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] > 0$.

With $p > 1$, the following holds:

$$\left[\frac{U(\hat{v}-b)}{U'(\hat{v}-b)} - (\hat{v}-b)\right] - p\left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] > \left[\frac{U(\hat{v}-b)}{U'(\hat{v}-b)} - (\hat{v}-b)\right] - \left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] > 0.$$

Therefore, $\left[\frac{U(\hat{v}-b)}{U'(\hat{v}-b)} - (\hat{v}-b)\right] - p\left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] > 0$ for $\hat{v} > 0$ such that

$0 < b_{DH,A}(\hat{v}) = b_{DH,N}(\hat{v}) = b < b^*$.

On the other hand, suppose $\hat{v} > 0$ is such that $b_{DH,A}(\hat{v}) = b_{DH,N}(\hat{v}) = b > b^*$, then

$\left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] < 0$ which combined with the fact that $\left[\frac{U(\hat{v}-b)}{U'(\hat{v}-b)} - (\hat{v}-b)\right] > 0$ gives us the

following: $\left[\frac{U(\hat{v}-b)}{U'(\hat{v}-b)} - (\hat{v}-b)\right] - p\left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] > 0$ for $\hat{v} > 0$ such that

$b_{DH,A}(\hat{v}) = b_{DH,N}(\hat{v}) = b > b^*$.

Therefore $\left[\frac{U(\hat{v}-b)}{U'(\hat{v}-b)} - (\hat{v}-b)\right] - p\left[\frac{U(\hat{v})}{U'(\hat{v}-b)} - \hat{v}\right] > 0$ for $\hat{v} > 0$ such that

$b_{DH,A}(\hat{v}) = b_{DH,N}(\hat{v}) = b$. Note that the inequality holds by strict concavity for $\hat{v} > 0$ such that

$b_{DH,A}(\hat{v}) > b_{DH,N}(\hat{v}) = b = b^*$.

Rearranging terms implies

$$\left[\frac{U(\hat{v}-b) - pU(\hat{v})}{U'(\hat{v}-b)}\right] > \hat{v} - b - p\hat{v} \text{ for } \hat{v} > 0. \tag{A16}$$

By properties of density and distribution functions, $\frac{f(v)}{F(v)} > 0$. In turn,

$$b'_{DH,A}(\hat{v}) = \frac{f(\hat{v})}{F(\hat{v})}\left[\frac{U(\hat{v}-b) - pU(\hat{v})}{U'(\hat{v}-b)}\right] > \frac{f(\hat{v})}{F(\hat{v})}\left[\hat{v} - b - p\hat{v}\right] = b'_{DH,N}(\hat{v}).$$

So for $\hat{v} = \min\{v > 0 | b_{DH,A}(v) = b_{DH,N}(v) \geq 0\}$, we have $b'_{DH,A}(\hat{v}) > b'_{DH,N}(\hat{v})$.

Now we show that $b_{DH,A}(v) > b_{DH,N}(v)$ for all $v > 0$. Suppose not. Then there exists one $v > 0$ such that $b_{DH,A}(v) \leq b_{DH,N}(v)$. By continuity, there exists $v > 0$ such that $b_{DH,A}(v) = b_{DH,N}(v)$. Recall the definition of $\hat{v}$, $\hat{v} = \min\{v > 0 | b_{DH,A}(v) = b_{DH,N}(v) \geq 0\}$. From above we have $b'_{DH,A}(\hat{v}) > b'_{DH,N}(\hat{v})$ so that $b_{DH,A}(v) < b_{DH,N}(v)$ for $v < \hat{v}$ in a neighborhood of $\hat{v}$ but then $b_{DH,A}(v) < b_{DH,N}(v)$ for all $v \in (0, \hat{v})$. Otherwise $b_{DH,A}(v)$ would cross $b_{DH,N}(v)$ at some $v < \hat{v}$ which contradicts the definition of $\hat{v}$. Therefore, for all $v < \hat{v}$, $b'_{DH,A}(v) > b'_{DH,N}(v)$ but

$b_{DH,A}(0)=b_{DH,N}(0)$ so $b_{DH,A}(v)>b_{DH,N}(v)$ in a neighborhood of zero. This contradicts $b_{DH,A}(v)<b_{DH,N}(v)$ for all $v<\hat{v}$ and proves $b_{DH,A}(v)>b_{DH,N}(v)$ for all $v>0$.

<u>Proof of Proposition 6</u>:

The equality follows directly from Propositions 1 and 3 and from Harris and Raviv's [1981] Theorem 6 (pp. 1492-1493). To prove the inequality, consider the expected revenue for the discriminative hybrid with risk-averse bidders:

$$\begin{aligned}E(R_{DH,A}) &= \sum_{j=N-Q+1}^{N} E(b_{DH,A}(v_j)) \\ &= \sum_{j=N-Q+1}^{N} \int_0^{\bar{v}} b_{DH,A}(v)\cdot h_j(v)dv\end{aligned}.$$

Because $b_{DH,A}(0)=b_{DH,N}(0)=0$ and by Proposition 2 $b_{DH,A}(v_j)>b_{DH,N}(v_j)\forall v_j>0$, we have $E(R_{DH,N})<E(R_{DH,A})$.

<u>Additional revenue hypothesis:</u>

Proposition A1: $E(R_{UH,N})=\left(1-\frac{S-Q}{N-Q}\right)E(R_U)<E(R_U)$

Proof:

Expected revenue in the uniform price hybrid with risk neutral bidders is given by:

$$\begin{aligned}E(R_{UH,N}) &= Q\cdot\left(1-\tfrac{S-Q}{N-Q}\right)E(v_{N-Q}) \\ &= \left(1-\tfrac{S-Q}{N-Q}\right)E(R_U)\end{aligned} \qquad \text{(A17)}$$

where $E(R_U)=Q\cdot E(v_{N-Q})$ by equation 2.2 of Cox, Smith, and Walker [1985]. The inequality follows from $\left(1-\frac{S-Q}{N-Q}\right)<1$.

Proposition A2: $E(R_{DH,N})=\left(1-\frac{S-Q}{N-Q}\right)E(R_{D,N})<E(R_{D,N})$

Proof:

Expected revenue in the discriminative hybrid with risk neutral bidders is given by:

$$\begin{aligned}E(R_{DH,N}) &= \sum_{j=N-Q+1}^{N} E(b_{DH,N}(v_j)) \\ &= \sum_{j=N-Q+1}^{N} \int_0^{\bar{v}} b_{DH,N}(v)\cdot h_j(v)dv\end{aligned} \qquad \text{(A18)}$$

where h_j is the density function of the *j*th order statistic in a sample size of *N*.

Substituting for $b_{DH,N}(v_j)$ using equation (9) in the text yields

$$E\left(R_{DH,N}\right)=\sum_{j=N-Q+1}^{N}\int_0^{\bar{v}}\left(1-\tfrac{S-Q}{N-Q}\right)b_{D,N}(v)\cdot h_j(v)dv$$
$$=\left(1-\tfrac{S-Q}{N-Q}\right)\sum_{j=N-Q+1}^{N}\int_0^{\bar{v}}b_{D,N}(v)\cdot h_j(v)dv \qquad \text{(A19)}$$
$$=\left(1-\tfrac{S-Q}{N-Q}\right)E\left(R_{D,N}\right)$$

where the last equality follows from equation (14) of Harris and Raviv [1981].

Proposition A3: $E\left(R_{UH,N}\right)<E\left(R_{UH,A}\right)<E\left(R_U\right)$

Proof:

Prove the second inequality first. Expected revenue in the uniform price hybrid with risk averse bidders is given by

$$E\left(R_{UH,A}\right)=Q\cdot E\left(\tilde{b}_{UH,A}\right)$$

where $\tilde{b}_{UH,A}$ represents the first rejected bid in the uniform price hybrid with risk averse bidders or the $(N-Q)^{th}$ bid. Expected revenue in the uniform price auction is given by $E\left(R_U\right)=Q\cdot E\left(v_{N-Q}\right)$. By Proposition 2, $\tilde{b}_{UH,A}<v_{N-Q}$ which implies $E\left(R_{UH,A}\right)<E\left(R_U\right)$. A similar logic proves the first inequality.

<u>Example with heterogeneous risk preferences: CRRA utility and the uniform price hybrid mechanism</u>

The equilibrium bid function for the uniform price hybrid with CRRA utility and heterogeneous risk preferences is given by $b_i=\left(1-\left(\frac{S-Q}{N-Q}\right)^{\frac{1}{1-r_i}}\right)v_i$ where v_i and r_i denote bidder *i*'s value and coefficient of relative risk aversion respectively. Consider two bidders with $v_1=v$, $r_1=r+\varepsilon$, $v_2=v+\gamma$, and $r_2=r$. Assume $0<r+\varepsilon<1$ and $\varepsilon>0$ so that both bidders are risk averse but bidder 1 is relatively more so. An efficient mechanism would guarantee that in equilibrium bidder 2 (bidder 1) outbids bidder 1 (bidder 2) provided $\gamma>0$ ($\gamma<0$). We proceed by solving for the value of γ for which $b_1>b_2$. Let $\kappa\equiv\frac{v}{1-p^{\frac{1}{1-r}}}\left(p^{\frac{1}{1-r}}-p^{\frac{1}{1-(r+\varepsilon)}}\right)>0$ where $p=\frac{S-Q}{N-Q}$.

Consider the following three ranges of possible values for γ: $\gamma<0$, $0<\gamma<\kappa$, and $\gamma>\kappa$. For values of γ less than zero or greater than κ, the bidder with the higher value submits the higher bid and the uniform price hybrid is efficient. However, when $0<\gamma<\kappa$, $v_2>v_1$ but $b_2<b_1$; under the uniform price hybrid, bidder 1 is more likely to win a unit of the good even though bidder 2 has a higher value for the good. Therefore, the uniform price hybrid is not in general efficient under heterogeneous CRRA risk preferences.